We have

$$|AD| + |EC| - |AC| = \frac{bc}{a+b} + \frac{ab}{b+c} - b$$
$$= b\left(\frac{c(b+c)+a(a+b)-(a+b)(b+c)}{(a+b)(b+c)}\right).$$

By the Law of Cosines $a^2+c^2-b^2 = 2ac\cos B$, so the equation above can be rewritten as

$$|AD| + |EC| - |AC| = \frac{2abc\left(\cos(B)-\frac{1}{2}\right)}{(a+b)(b+c)}.$$

Hence,

$$\begin{aligned} |AD| + |EC| = |AC| &\iff \angle ABC = 60^\circ, \\ |AD| + |EC| > |AC| &\iff \angle ABC < 60^\circ, \\ |AD| + |EC| < |AC| &\iff \angle ABC > 60^\circ. \end{aligned}$$

Next we turn to solutions from our readers to problems of the 2005 Vietnam Mathematical Olympiad given at [2008 : 81].

1. Find the smallest and largest values of the expression $P = x + y$, where x and y are real numbers satisfying $x - 3\sqrt{x+1} = 3\sqrt{y+2} - y$.

Solved by Arkady Alt, San Jose, CA, USA; and Michel Bataille, Rouen, France. We give the solution of Bataille.

We show that $P_{\min} = \frac{1}{2}(9+3\sqrt{21})$ and $P_{\max} = 9+3\sqrt{15}$.

First, $x=-1$ and $y = \frac{1}{2}(11+3\sqrt{21})$ satisfy the constraint equation $x-3\sqrt{x+1} = 3\sqrt{y+2}-y$ (easily checked using $10+2\sqrt{21} = \left(\sqrt{3}+\sqrt{7}\right)^2$) and $P = \frac{1}{2}(9+3\sqrt{21})$. Similarly, for $x = \frac{1}{2}(10+3\sqrt{15})$, $y = \frac{1}{2}(8+3\sqrt{15})$, we have $P = 9+3\sqrt{15}$ and the constraint is satisfied. Now, let x and y satisfy the constraint equation. Then $P = 3\sqrt{x+1}+3\sqrt{y+2}$, so that

$$P^2 = 9\left(P+3+2\sqrt{(x+1)(y+2)}\right). \tag{1}$$

It follows that $P \geq 0$ and $P^2 - 9P - 27 \geq 0$. Thus, P is not less than the positive solution of the quadratic $x^2 - 9x - 27 = 0$ and we deduce that $P \geq \frac{1}{2}(9+3\sqrt{21})$. From the AM-GM Inequality and (1), we obtain

$$P^2 \leq 9(P+3+x+1+y+2) = 9(2P+6) = 18P+54,$$

or $P^2 - 18P - 54 \leq 0$, which implies that $P \leq 9+3\sqrt{15}$. The proof is complete.